Transport properties of clean and disordered Josephson-junction arrays

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We investigate the influence of quantum fluctuations and weak disorder on the vortex dynamics in a two-dimensional superconducting Berezinskii-Kosterlitz-Thouless system. The temperature below which quantum fluctuations dominate the vortex creep is determined and the transport in this quantum regime is described. The crossover from quantum to classical regime is discussed and the quantum correction to the classical current-voltage relation is determined. It is found that weak disorder can effectively reduce the critical current as compared to that in the clean system.

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The physics of the Berezinskii-Kosterlitz-Thouless (BKT) transition in thin superconducting films is re-emerging as one of the mainstreams in current condensed-matter physics. The interest is motivated by recent advances in studies of layered high-temperature superconductors,[1,2] the discovery of the superconductivity at the interface between the insulating oxides,[3,4] studies in thin superconducting films uncovering the role of the two-dimensional (2D) superconducting fluctuations,[5,6] and the intense developments in the physics of the superconductor-insulator transition where the BKT transition may play a major role.[7]

The predicted benchmark of the transition that serves to detect it experimentally is the change in the shape of the *I*-*V* characteristics, $V \propto I^{1+\alpha}$ from $\alpha=0$ above the transition, $T > T_{\mathrm{BKT}}$, to $\alpha=2(T_{\mathrm{BKT}}/T)$ at $T \leq T_{\mathrm{BKT}}$.[8,9] However, experimental data on superconducting films show appreciable deviations from the theoretical predictions and are still inconclusive.[10] Among the possible sources of the deviation from the classic predictions, one can consider the finite-size effect,[11,12] and effects of disorder.[13–15] Another important issue is the role of quantum effects which become crucial when the BKT transition occurs at low enough temperatures. In this Brief Report we will analyze the role of quantum effects in the BKT transition paying a special attention to the intermediate region of the interplay between thermal and quantum contributions. We will discuss the effect of disorder-generated vortices on the BKT transition, neglecting quantum fluctuations, namely, the effective reduction in the critical current as compared to that in clean samples.

Model. We choose a disordered Josephson junction array as a convenient discrete model for the 2D disordered superconducting film.[16] The Hamiltonian describing the system is

$$\mathcal{H} = \frac{1}{2}\sum_{i,j}(C^{-1})_{i,j}\hat{n}_i\hat{n}_j - J\sum_{\langle i,j\rangle}\cos(\hat{\varphi}_i - \hat{\varphi}_j - A_{ij}), \tag{1}$$

where $[\hat{n}_j, \hat{\varphi}_k] = -2ei\delta_{j,k}$. We ignore single-electron tunneling and other sources of dissipation. The only nonvanishing elements of the capacitance matrix C_{ij} are its diagonal elements $C_{jj}=4C$ (no summation over the repeated index) and $C_{ij} = -C$ for the nearest neighbors i,j, i.e., the capacitance to the ground is assumed negligible as compared to the mutual capacitances of the superconducting islands. The second sum in Eq. (1) is taken over all nearest-neighbor pairs on a square lattice. Random-phase shifts A_{ij} result from the deviations of the flux in a distorted plaquette from an integer multiple of the flux quantum $\Phi_0 = \hbar c/2e$.[17]

In the clean classical case, i.e., for $A_{ij}=0$ and in the limit $C \to \infty$, the physics of the system can be most adequately described in terms of vortices that experience the superconducting BKT transition at the temperature $T_{\mathrm{BKT}} \simeq \pi\tilde{J}/2$, where $\tilde{J}$ denotes the renormalized coupling constant. It is convenient to decompose the phase at the site i, as $\varphi_i = \varphi_i^{(v)} + \varphi_i^{(sw)}$ where (v) and (sw) stand for the vortex and the spin wave part, respectively. Then, the vortex Hamiltonian can be written as

$$\mathcal{H}_v = -J\pi\sum_{i\neq j} m_i m_j \ln\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi} + \sum_i E_c m_i^2, \tag{2}$$

where E_c denotes the core energy of a vortex. The sums are taken over the sites $\mathbf{r}_i$ of a dual lattice; m_i is the vorticity of the ith vortex, and we assumed that $\Sigma_i m_i = 0$, where ξ denotes the superconductor coherence length.

Next we want to include quantum fluctuations. After going over to the path integral representation of the partition function and integrating out the charge degrees of freedom, the action of the Josephson junction array in the limit $E_c = e^2/2C \ll J$ assumes the form[16,18]

$$S = \int d\tau\left[\frac{M}{2}\sum_i(\partial_\tau \mathbf{r}_i)^2 + \mathcal{H}_v\right]. \tag{3}$$

The vectors $\mathbf{r}_i(\tau)$ are the world lines of the vortices and $M = h^2C/(8e^2\xi^2)$.

Clean case. We begin with the discussion of a clean case. If we apply an external transport current, it will exert the force $\mathbf{f} \sim \mathbf{j}$ on the vortices, where $\mathbf{j}$ is the current density.[19] This generates an additional term $-\Sigma_i m_i \mathbf{f}\cdot\mathbf{r}_i$ in Eq. (2). In order to describe the effect of vortices on the current-voltage relation quantitatively, we consider the effect of vortices crossing the system transversely to the transport current. This motion dissipates energy. The Bardeen-Stephen flux flow resistance[20] gives for the current-voltage (*V*-*I*) relation

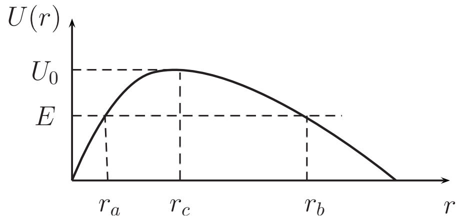

FIG. 1. Potential barrier for the separation of the vortex-antivortex pair.

$$V = 2\pi\xi^2\rho_n n_v I, \tag{4}$$

where n_v is the vortex density and ρ_n is the normal-state resistivity. The rate equation for the vortex density is

$$\partial_t n_v = \Gamma - \frac{\xi^2}{\tau_{rec}} n_v^2. \tag{5}$$

Here Γ denotes the rate of generation of free vortices, while the second term on the right-hand side of Eq. (5) describes their recombination, τ_{rec} denotes the recombination parameter. The steady-state value $n_v = (\tau_{rec}\Gamma)^{1/2}/\xi$ of the vortex density determines the current-voltage relation.

In order to determine Γ, we consider the appearance of a vortex-antivortex pair and its subsequent separation via tunneling or thermal activation under the influence of the external force $\mathbf{f}$. In the clean case this process is symmetric, i.e., the coordinates of the vortex $\mathbf{r}_1$ and the antivortex $\mathbf{r}_2$ satisfy $\mathbf{r}_1 = -\mathbf{r}_2 = \mathbf{r}$ with $\mathbf{f}\cdot\mathbf{r} = fr$. The action of the vortex pair can be rewritten as

$$S = \int d\tau [M(\partial_\tau r)^2 + U(r)], \tag{6}$$

where $U(r) = 2\pi J \ln(\frac{2r}{\xi}) - 2fr + 2E_c$. The problem effectively reduces to a single-particle motion through one-dimensional potential barrier $U(r)$.[21]

The rate Γ is given by[22]

$$\Gamma \sim \int_0^\infty dE\, \Gamma(E) e^{-E/T}, \tag{7}$$

where $\Gamma(E)$ denotes the zero-temperature tunneling rate of a particle in the potential $U(r)$ having an energy E. For low temperatures and hence E smaller than the barrier height $U_0 = 2\pi J[\ln(\frac{2\pi J}{f\xi}) - 1] + 2E_c$, $\Gamma(E)$ in the Wentzel-Kramers-Brillouin approximation is

$$\Gamma(E) = e^{-4\sqrt{M}\int_{r_a(E)}^{r_b(E)} dr \sqrt{U(r)-E}/\hbar}, \tag{8}$$

where $r_{a/b}(E)$ satisfy $U(r_{a/b}) = E$ (see Fig. 1). In the following different regimes will be considered.

(i) At zero temperature the only contribution in Eq. (7) comes from $E=0$. The generated voltage for small currents $(f\xi/J \ll 1)$ is

$$V \sim \Gamma^{1/2} \sim e^{-S(0,0)/2\hbar},$$

$$\frac{S(0,0)}{\hbar} \approx c_1 \frac{\sqrt{M}(2J\pi)^{3/2}}{\hbar f}\left(\ln\frac{2J\pi}{f\xi}\right)^{3/2}. \tag{9}$$

c_1 is a positive constant of the order of unity and

$$\frac{S(E,T)}{\hbar} = \frac{E}{T} + 4\sqrt{M}\int_{r_a(E)}^{r_b(E)} dr \frac{\sqrt{U(r)-E}}{\hbar} \tag{10}$$

is the action of the classical path of the particle in the potential $-U(r)$ with the energy E and mass $2M$. The result, Eq. (9), is in agreement with that of Ref. 23 where it is obtained using the different technique.[24] We find that the result, Eq. (9), holds also at finite temperatures as long as

$$T \le T_0 = \frac{1}{c_2}\frac{\hbar f}{\sqrt{\pi 2JM}}\frac{1}{\sqrt{\ln\frac{\pi 2J}{f\xi}}}, \tag{11}$$

where c_2 is positive constant of the order of unity.

(ii) At intermediate temperatures $T_0 < T < T^*$, where

$$T^* = \frac{\hbar}{2\pi}\sqrt{\frac{-U''(r_c)}{2M}} = \frac{\hbar f}{2\pi}\sqrt{\frac{1}{MJ\pi}}, \tag{12}$$

the main contribution in Eq. (7) comes from the stationary point E_T. Therefore, $V \sim \exp[-S(E_T,T)/2\hbar]$. E_T depends on the temperature and is implicitly given by the equation

$$\frac{\hbar}{T} = 2\sqrt{M}\int_{r_a(E_T)}^{r_b(E_T)} dr \frac{1}{\sqrt{U(r)-E_T}} = \tau(E_T), \tag{13}$$

where $\tau(E)$ can be interpreted as the period of the classical motion of a particle with the mass $2M$ and energy E, in the potential $-U(r)$. Since $\tau(E)$ is the monotonically decreasing function of E for small currents, Eq. (13) has the unique solution E_T for every T in a range $T_0 \le T \le T^*$. We come back to the discussion of the voltage characteristic in this regime below.

(iii) At even higher temperatures $T^* < T \le T_{\mathrm{BKT}}$, the decay rate is dominated by $E > U_0$ (Refs. 22 and 25) and the thermally activated breaking of vortex pairs dominates the dynamics. Then, the decay rate is given by the Arrhenius law $\Gamma \sim \exp[-S_{class}/\hbar]$, where $S_{class} = \hbar U_0/T$. The voltage-current relation reads[8,26]

$$V \sim fe^{-U_0/(2T)} \sim j^{\delta(T)}, \quad \delta(T) = 1 + \pi J/T. \tag{14}$$

Taking into account the presence of other vortices by replacing $J \to \tilde{J}$, the coefficient assumes a universal value $\delta(T_{\mathrm{BKT}}) = 3$.

(iv) At $T > T_{\mathrm{BKT}}$ a finite density of free vortices appears in an equilibrium, and the system is characterized by a linear current-voltage relation for small enough currents.

Next, we consider crossover from the quantum $(T \le T_0)$ to the classical regime $(T > T^*)$ in more detail. Within the semiclassical approximation the decay rate is given, with the exponential accuracy, by $\Gamma \sim \exp[-S_{\min}/\hbar]$, where $S_{\min}$ is the action of the trajectory minimizing the Euclidean action of Eq. (6). For temperatures below T_0 the extremal action is $S_{\min} = S(0,0)$, in the intermediate region $(T_0 < T < T^*)$ the minimal action is $S_{\min} = S(E_T,T)$, and in the high-temperature

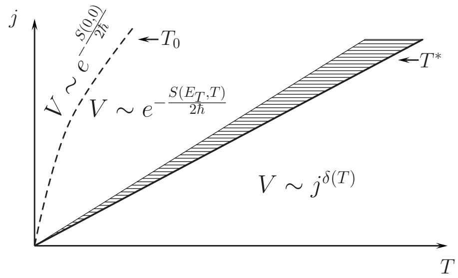

FIG. 2. Dynamic phase diagram in current-temperature coordinates showing different types $V(j,T)$ dependencies for $T<T_{\mathrm{BKT}}$. The dashed and the solid lines sketch $T_0(j)$ and $T^*(j)$, respectively. In the domain $T<T_0$ quantum tunneling of vortices dominates the vortex dynamics while at $T>T^*$ the voltage-current characteristics is determined by the thermally activated motion. In the shaded region the quantum correction to the classical result, given by Eq. (17), applies.

regime the trajectory extremizing the action is time independent, and therefore $S_{\min}=\hbar U_0/T$. We find that $S_{\min}$ at T^* has a continuous first derivative with respect to temperature while the second derivative has a jump

$$\left.\frac{dS(E_T,T)}{dT}\right|_{T^*} = \left.\frac{dS_{\mathrm{class}}}{dT}\right|_{T^*},$$

$$\left.\frac{d^2S(E_T,T)}{dT^2}\right|_{T^*} < \left.\frac{d^2S_{\mathrm{class}}}{dT^2}\right|_{T^*}. \tag{15}$$

Following Ref. 27 we call this situation a second-order transition at the crossover point.[28] The result of Eq. (15) is a general property of a massive particle trapped in a metastable state formed by a potential $U(r)$, provided $\tau(E)$ is a monotonously decreasing function of energy.[29]

Generally, in the case of a second-order transition the trajectory extremizing the action can be written as[27]

$$r(\tau) = r_c + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi T}{\hbar} n\tau\right), \tag{16}$$

where the coefficients $|a_n| \ll |a_1|$ $(n>1)$ are small near the transition temperature T^*. Substituting $r(\tau)$ in Eq. (6), the action can be expanded in powers of a_n, yielding an effective action $S \approx U_0\hbar/T+\alpha a_1^2+\beta a_1^4$, where the coefficient α is negative below T^* and vanishes at the transition temperature T^*.[27] Then the coefficient a_1 can be found from the minimization of the action S and the minimal action is $S_{\min} = U_0\hbar/T-\alpha^2/(4\beta)$. Following Ref. 27, we determine the coefficients α and β and find a quantum correction to the classical result of Eq. (14)

$$V \sim j^{\delta(T)} e^{\Delta},$$

$$\Delta = \frac{(T^2-T^{*2})^2}{TT^{*3}} \frac{\sqrt{MJ^3}}{\hbar f} \frac{\pi^{5/2}}{1+2[1-4(T/T^*)^2]^{-1}}. \tag{17}$$

This result is valid near the transition, for temperatures approaching T^* from below, see Fig. 2. We conclude that quantum effects significantly enhance the decay rate in comparison to the classical rate for the asymptotically small currents. It would be interesting to probe the result of Eq. (17) in experiments.

Disordered case. Next we include disorder into the consideration in the limit $C\to\infty$. The phase shifts are assumed to be uncorrelated from bond to bond, and each is Gaussian distributed with the mean value and the variance

$$\overline{A_{ij}} = 0, \quad \overline{A_{ij}^2} = \sigma, \tag{18}$$

respectively. Then, an additional term $\Sigma_i m_i V(\mathbf{r}_i)$ is generated in Eq. (2), where $V(\mathbf{r}_i)=2\pi J\Sigma_j Q_j \ln(|\mathbf{r}_i-\mathbf{r}_j|/\xi)$. $Q_i = (1/2\pi)\Sigma_{\langle plaq\rangle} A_{ij}$ are the frozen charges sitting on the dual lattice in the center of a plaquette whereas the sum is over a plaquette formed by four bonds. From Eq. (18) follows $\overline{[V(\mathbf{r}_i)-V(\mathbf{r}_j)]^2}=4\pi\sigma J^2 \ln(|\mathbf{r}_i-\mathbf{r}_j|/\xi)$.

It was shown in Ref. 13, that the system in the classical case, at $T=0$ undergoes a disorder-driven transition from the "ordered" BKT state to a disordered phase at the critical disorder strength $\sigma_c=\pi/8$. In the ordered BKT phase vortices appear, on average, only in a form of the bound pairs. Indeed, the energy of a vortex pair with the separation R and $m_1=-m_2=1$ in a clean sample is given by $2\pi J \ln(R/\xi)$. Since $\overline{V(\mathbf{r}_i)}$ is Gaussian distributed, the typical energy gain is $-2J\sqrt{\pi\sigma}\ln(R/\xi)$ which is smaller by a factor $\sim[\ln(R/\xi)]^{-1/2}$ than the energy cost of a pair. However, the maximum energy gain of a vortex dipole in a region of linear size $L>R$ is larger by a factor $\sqrt{2\ln N}$ than the typical energy gain, which arises from the N independent realizations of the vortex positions.[14] The disorder potential, that one vortex-antivortex pair of size $R+dR$ feels, is uncorrelated when the pair is translated over a distance larger than R.[30] Therefore, we introduce a lattice with a lattice constant R. Since also the correlations of disorder potential inside the cell give only subleading-order corrections,[30] we estimate $N\approx(L/R)^2(R/\xi)^2(2\pi R/\xi)dR/\xi$. For $dR\approx R$, we get the free energy of the pair at $T=0$

$$E \approx 2\pi J \ln\frac{R}{\xi}\left[1-\sqrt{\frac{4\sigma}{\pi}\frac{\ln(LR/\xi^2)}{\ln(R/\xi)}}\right]. \tag{19}$$

Thus, if $R\approx L$, the total energy of the corresponding vortex pair becomes negative and free vortices are favored by disorder provided $\sigma>\sigma_c=\pi/8$, in an agreement with the renormalization group result in Ref. 13. Note that strictly speaking these vortices are "pseudofree" since despite the fact that their attraction is overruled by disorder, they remain pinned by the same disorder-induced forces. It follows from the above reasoning that even for $\sigma<\sigma_c$ some rare vortex pairs of the negative energy can appear. From Eq. (19) we get that their maximal size is $R_c\approx\xi(L/\xi)^{1/(2\sigma_c/\sigma-1)}$, which reaches the size of the system for $\sigma\to\sigma_c-0$, as expected. Typically there is a single dipole of the size R_c in the system. If we divide the system into M^2 subsystems, each part will contain a dipole of the maximum size $R_M\approx R_c M^{-1/(2\sigma_c/\sigma-1)}$. The density of dipoles of the size R_M is $\xi^{-2}(R_M/\xi)^{2(1-2\sigma_c/\sigma)}$ at $T=0$, in agreement with Ref. 30.

We further determine the critical current. If the transport current is strong enough, it will depin vortices such that the dissipation sets in. A crude estimate for the critical depinning force at $T=0$ and $\sigma<\sigma_c$ is given by

$$f_c \sim \frac{J}{R_c} \sim \frac{J}{\xi}\left(\frac{L}{\xi}\right)^{-1/(2\sigma_c/\sigma-1)} \tag{20}$$

since smaller dipoles are depinned at larger forces. The influence of disorder on the voltage-current relation is left for further studies.

Conclusion. We have investigated transport properties of Josephson junction arrays taking into account the influence of quantum fluctuations on the unbinding of vortex pairs for $E_c \ll J$. At sufficiently low temperatures the quantum tunneling of vortices turns out to be more probable than their thermal activation. We have derived the *V-I* relation corresponding to the quantum creep of the BKT vortices and found the range of temperatures, $0 \le T \le T_0$, where this law is applicable. We have determined the temperature T^* above which the thermally activated breaking of vortex pairs dominates the vortex nucleation. We have discussed the region of intermediate temperatures $T_0 < T < T^*$ where a crossover from classical to quantum behavior occurs and found the quantum correction to the classical result, see Eq. (17). The results are schematically summarized in Fig. 2 and can be straightforwardly extended to the quantum limit $E_c \gg J$, where the transport is mediated by the motion of charges dual to the superconducting vortices, via the standard dual transformation. Moreover, in the presence of positional disorder and for $C\to\infty$, we have shown that additional vortices generated by the disorder contribute to transport, effectively reducing the critical current as compared to that in a clean system.

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[1] A. Rüfenacht, J.-P. Locquet, J. Fompeyrine, D. Caimi, and P. Martinoli, Phys. Rev. Lett. **96**, 227002 (2006).
[2] D. Matthey, N. Reyren, J.-M. Triscone, and T. Schneider, Phys. Rev. Lett. **98**, 057002 (2007).
[3] N. Reyren, S. Thiel, A. D. Caviglia, L. Fitting Kourkoutis, G. Hammerl, C. Richter, C. W. Schneider, T. Kopp, A.-S. Rüetschi, D. Jaccard, M. Gabay, D. A. Muller, J.-M. Triscone, and J. Mannhart, Science **317**, 1196 (2007).
[4] A. D. Caviglia, S. Gariglio, N. Reyren, D. Jaccard, T. Schneider, M. Gabay, S. Thiel, G. Hammerl, J. Mannhart, and J.-M. Triscone, Nature (London) **456**, 624 (2008).
[5] R. W. Crane, N. P. Armitage, A. Johansson, G. Sambandamurthy, D. Shahar, and G. Grüner, Phys. Rev. B **75**, 094506 (2007).
[6] A. Pourret, H. Aubin, J. Lesueur, C. A. Marrache-Kikuchi, L. Bergé, L. Dumoulin, and K. Behnia, Nat. Phys. **2**, 683 (2006).
[7] V. M. Vinokur, T. I. Baturina, M. V. Fistul, A. Yu. Mironov, M. R. Baklanov, and C. Strunk, Nature (London) **452**, 613 (2008).
[8] B. I. Halperin and D. R. Nelson, J. Low Temp. Phys. **36**, 599 (1979).
[9] P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).
[10] J. M. Repaci, C. Kwon, Q. Li, X. Jiang, T. Venkatessan, R. E. Glover III, C. J. Lobb, and R. S. Newrock, Phys. Rev. B **54**, R9674 (1996); D. R. Strachan, C. J. Lobb, and R. S. Newrock, *ibid.* **67**, 174517 (2003); M. M. Özer, J. R. Thompson, and H. H. Weitering, *ibid.* **74**, 235427 (2006); M. M. Ozer, Y. Jia, Z. Zhang, J. R. Thompson, and H. H. Weitering, Science **316**, 1594 (2007); F. Tafuri, J. R. Kirtley, D. Born, D. Stornaiuolo, P. G. Medaglia, P. Orgiani, G. Balestrino, and V. G. Kogan, Europhys. Lett. **73**, 948 (2006).
[11] V. G. Kogan, Phys. Rev. B **75**, 064514 (2007).
[12] A. Gurevich and V. M. Vinokur, Phys. Rev. Lett. **100**, 227007 (2008).
[13] T. Nattermann, S. Scheidl, S. E. Korshunov, and M. S. Li, J. Phys. I (France) **5**, 565 (1995).
[14] S. E. Korshunov and T. Nattermann, Phys. Rev. B **53**, 2746 (1996); Physica B **222**, 280 (1996).
[15] L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. B **80**, 214506 (2009).
[16] U. Eckern and A. Schmid, Phys. Rev. B **39**, 6441 (1989).
[17] E. Granato and J. M. Kosterlitz, Phys. Rev. B **33**, 6533 (1986).
[18] R. Fazio and G. Schön, Phys. Rev. B **43**, 5307 (1991).
[19] M. Tinkham, *Introduction to Superconductivity* (Dover, New York, 2004).
[20] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).
[21] Note that the fact that the system is two-dimensional is important for pre-exponential factors in the vortex production rate Γ. However, in the following we will determine Γ with an exponential accuracy since we are interested in asymptotically small currents.
[22] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).
[23] R. Iengo and G. Jug, Phys. Rev. B **54**, 13207 (1996).
[24] I. P. Ao, J. Low Temp. Phys. **89**, 543 (1992), the same leading dependence on f is found but a different logarithmic dependence on f.
[25] V. I. Goldanskii, Dokl. Akad. Nauk SSSR **124**, 1261 (1959) [Sov. Phys. Dokl. **4**, 74 (1959)].
[26] S. Doniach and B. A. Huberman, Phys. Rev. Lett. **42**, 1169 (1979).
[27] A. I. Larkin and Yu. N. Ovchinnikov, JETP Lett. **37**, 382 (1983); Sov. Phys. JETP **59**, 2 (1984).
[28] Note that quantum fluctuations smear the transition in a narrow temperature region near T^* (Ref. 29).
[29] E. M. Chudnovsky, Phys. Rev. A **46**, 8011 (1992).
[30] L.-H. Tang, Phys. Rev. B **54**, 3350 (1996).